Scaling laws in the cosmic structure and renormalization group

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Abstract
There is an evidence of a scale-invariant matter distribution up to scales over 10 Mpc. We review scaling (fractal or multifractal) models of large-scale structure and their observational evidence. We conclude that the dynamics of cosmological structure formation seems to be driven to a multifractal attractor. This supports previous studies, which we review, of structure formation by means of the renormalization group within a hydrodynamic formulation.

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1. Introduction

Both the old idea of hierarchical clustering in the universe and the statistical analysis of the distribution of galaxies have led to scaling laws in the cosmic structure and fractal models of it [1]. Scaling is usually demonstrated by the appearance of power laws in the correlation functions. There are other scaling laws in the cosmic structure, referring to other features. For example, the counterpart of galaxy clusters are galaxy *voids*, namely, large empty regions in the galaxy distribution. Fractal voids have scaling properties in the rank-ordering of voids [2], verified in galaxy surveys [3].

Nevertheless, a consensus on the range of application of scaling laws in the description of the cosmic structure has not been reached (a recent discussion is in [4]). The most general scaling model is the multifractal model, introduced in cosmology to describe 'non-uniform' fractal clustering [5]. However, the study of cosmological *N*-body simulations has led to halo models of large-scale structure [6] which do not assume any scaling laws. Fortunately, multifractal models can be formulated in a way that closely resembles usual halo models, allowing us to unveil scaling laws in *N*-body simulations [7].

Further to the description of cosmic structure, the problem of structure formation has given rise to scaling laws with dynamical content. For example, a popular model of structure formation, the adhesion model [8, 9], displays dynamical scaling and, in addition, leads to

the multifractal structure [10]. Structure formation is a nonlinear process, like other *non-equilibrium* processes in statistical mechanics that have scale invariance. Scaling is crucial in critical phenomena, which are phenomena that take place in equilibrium statistical mechanics for definite values of the parameters. The renormalization group (RG), which arose in quantum field theory, was soon applied to critical phenomena [11] and, more recently, it has also been applied to non-equilibrium processes (dynamical RG). The latter application can be extended to the process of structure formation in cosmology. However, the dynamical RG approach is perturbative and encounters some problems in this regard.

A basic feature of modern formulations of the RG is that they deal with many-body systems by progressively removing irrelevant degrees of freedom until the 'renormalized system' becomes manageable. The coarse-graining procedure, widely used in statistical mechanics and hydrodynamics, is inspired in the same idea. It is intrinsically non-perturbative and can be applied to structure formation.

We begin by reviewing the evidence of scaling in the large-scale structure of matter, arising from observational data or cosmological N-body simulations. Our analysis is consistent with the hypothesis that the dynamics of structure formation is driven to a *multifractal attractor*, like other non-equilibrium processes in statistical mechanics. This result justifies the hypothesis of *dynamical* scaling and, in particular, the dynamical RG approach. We connect with work along this line within the adhesion model, which is quite successful in one dimension, but seems to require additional ingredients in three dimensions. Therefore, we turn to approaches based on the coarse-graining procedure and, in particular, we formulate the coarse-graining 'exact' RG group for structure formation.

Since we assume the existence of a multifractal attractor, we will not review approaches based on perturbation theory about the linear regime. Here we just mention the interesting recent application of the RG by Crocce and Scoccimarro [12].

2. Scaling laws in the large-scale structure

2.1. Scaling of galaxy clustering

Hierarchical clustering consists of a hierarchy of clusters of clusters. In general, fractal geometry studies sets (or functions) that are irregular (non-smooth) and have *fine structure*, namely, detail at all scales. Usually, the fine structure of a fractal is due to its *self-similarity*, that is, to the set being similar to parts of itself, in a strict or approximate sense. Random fractals only have *statistical* self-similarity, which implies that the correlation functions are power laws.

A useful description of random fractals is through the *number–radius* relation, which expresses the number of points in a ball of radius r centred on one point and *averaged* over every point: it has to be the power law $N(r) = Br^D$, where D is the fractal dimension and B is a constant. $N(r)$ is the cumulative conditional probability, that is, the integral of the conditional probability $\Gamma(r)$, which measures the average probability of finding another point at distance r from one given point. In turn, $\Gamma(r)$ is directly related to the reduced two-point correlation function $\xi(r)$, namely, $\Gamma(r) = \bar{\rho}(1 + \xi(r))$, where $\bar{\rho}$ is the average density. Both $N(r)$ and $\Gamma(r)$ are used to test scaling: their log–log plots must be linear, with slopes D and $D - 3$, respectively.

The distribution must have a transition to homogeneity on very large scales, where $D \to 3$. The corresponding scale can be defined in terms of the non-dimensional correlation $\xi(r)$, by writing it as $\xi(r) = (r_0/r)^\gamma$, where $\gamma = 3 - D$ and r_0 is the scale of transition to homogeneity. We see that the strongly nonlinear regime, $\xi \gg 1$, is the fractal regime, where ξ, Γ *and* N

are all power laws. In the homogeneous regime, for $r \gg r_0$, $\xi \to 0$ and $\Gamma \to \bar{\rho}$, while $N \propto r^3$.

Typical values of the fractal dimension and the scale of transition to homogeneity are about $D \simeq 2$ [1, 13] and $r_0 \simeq 15$ Mpc h^{-1} [13], respectively.

2.1.1. Scaling of voids. Voids in the galaxy distribution scale if the number of voids with a given size is a power law of the size. It is again convenient to employ the cumulative count, namely, the number of voids $N(L > \ell)$ with linear size larger than a reference ℓ, which fulfills $N(L > \ell) \propto \ell^{-D}$, where D is the fractal dimension [2]. The cumulative count is the *rank*, so the preceding law can also be expressed as a power-law dependence of the size with the rank (if the size refers to volume, the exponent is $3/D$). Such type of dependence is an instance of *Zipf's law*.

The scaling of voids in galaxy surveys is still uncertain. Recent analyses find sets of convex-like voids that satisfy Zipf's law [3]. The fractal dimension deduced from them, $D \simeq 2$, coincides with the previously cited dimension deduced from clustering. However, $D = 2$ is the dimension of the boundary of voids and, therefore, the minimal value of D in the Zipf law for voids [14]. The actual fractal dimensions of the samples may be smaller [3].

2.1.2. Luminosity segregation. Different galaxy populations may have different statistical properties. If these populations are fractal, they may have different dimensions. In fact, although $D \simeq 2$ is typical, other analyses yield smaller values, and one can change D somewhat by selecting different galaxy populations: systematic analyses of galaxy populations in the Sloan Digital Sky Survey (SDSS) selected by luminosity show a decrease of fractal dimension with luminosity [15]. Thus, the galaxy distribution may be, rather than a simple fractal, a *multifractal*, in which various dimensions appear naturally.

2.2. Multifractal model

Multifractals are the *most general* scaling distributions. They appear frequently as attractors of nonlinear dynamical systems. Multifractal measures represent highly irregular mass distributions, that is, with mass concentrations of very different magnitude. This magnitude is defined by the *local* dimension $\alpha(\boldsymbol{x})$:

$$m[B(\boldsymbol{x}, r)] \sim r^{\alpha(\boldsymbol{x})}, \tag{1}$$

where $m[B(\boldsymbol{x}, r)]$ is the mass in the ball of radius r centred on $\boldsymbol{x}$. In a regular mass distribution $\alpha = 3$ (constant), so mass concentrations $\alpha(\boldsymbol{x}) < 3$ are singularities. On the other hand, an ordinary fractal can be considered endowed with a *uniform* mass distribution over it, such that $\alpha < 3$ is the *constant* fractal dimension. Thus, in the context of multifractals, ordinary fractals are called *monofractals* (or *unifractals*). Full-fledged multifractals possess a range of dimensions α, namely, $0 < \alpha_{\min} \leqslant \alpha \leqslant \alpha_{\max}$. Every set of points in which α takes a definite value is a fractal set. Therefore, a multifractal can be considered as a set of interwoven fractals with running α. The *multifractal spectrum* $f(\alpha)$ is the function that gives the dimension of the fractal with exponent α.

Statistical moments are defined by

$$M_q(r) = \int \mathrm{d}m(\boldsymbol{x}) m[B(\boldsymbol{x}, r)]^{q-1}. \tag{2}$$

M_1 is the total mass (normalized to one). The two-point correlation integral $M_2(r)$ is the continuous version of the number function $N(r)$. Multifractals are *singular* non-uniform

distributions, so moments with integer q are not sufficient; one has to consider the full set of moments $M_q(r)$ for $-\infty < q < \infty$. We can then define the function $\tau(q)$ that gives the scaling behaviour of this full set of moments,

$$M_q(r) \sim r^{\tau(q)}. \tag{3}$$

$\tau(q)$ determines the multifractal spectrum through a Legendre transform [5]: assuming $\alpha(q) = \tau'(q)$ to be monotone, $f(\alpha) = q(\alpha)\alpha - \tau[q(\alpha)]$.

2.2.1. The fractal distribution of halos. We associate halos with *singular* mass concentrations, namely, points with $\alpha(x) < 3$, such that the density given by equation (1) diverges as $r \to 0$. Note that scale invariance prevents us from assigning these singularities definite sizes or masses. Therefore, to properly define halos we must use some small coarse-graining scale L. In N-body simulations, the natural coarse-graining scale is the linear size of the volume per particle. Initially and during the linear evolution, there is one particle per volume element. So halos only arise in the nonlinear stage, as some volume elements concentrate particles from other regions that become *voids*.

Therefore, we identify halos with mass concentrations of size L in a multifractal. Since $\alpha \sim \log m / \log L$, every population formed by *equal-mass* halos is a monofractal, although different populations have different dimensions. We can describe this difference between populations as a kind of *bias*, albeit of *nonlinear* type. In a multifractal analysis of N-body simulations [7], we have found that populations of halos of given mass are fractals, with mass-dependent dimension. Their common scale of transition to homogeneity is $r_0 \simeq 14h^{-1}$ Mpc.

An interesting quantity is the *mass function* of halos, namely, the number of halos with a given mass. The Press-Schechter formalism predicts a power law (exponentially cut off at large mass) with exponent connected with the initial power spectrum [6]. This form is observed in our analysis [7], *but the power law has a fixed exponent, namely, $N(m) \sim m^{-2}$*, which corresponds to the spectral index $n = -3$ of the initial power spectrum, just beyond the allowed range. Moreover, the initial power spectra of the simulations we have analysed are not power laws, but the mass function is always the same, *independently of the initial conditions*.

3. Dynamical scaling

In the strongly nonlinear regime, when the initial condition is forgotten, dynamical scaling implies that a dynamical field $\varphi(x, t)$ satisfies a scaling relation $\varphi_L - \langle\varphi\rangle \sim L^\chi f(t/L^z)$, where φ_L is the field coarse-grained over a length L (see equation (7)), χ, z are critical exponents, $\lim_{u\to\infty} f(u) = 1$ and $\lim_{u\to 0} f(u) \sim u^{\chi/z}$. In words, the fluctuations grow with time as one power law and they eventually reach saturation, in which state they depend on L as another power law. Dynamical scaling is customary in the physics of surface growth and other non-equilibrium processes [16]. Similarly to the situation in static critical phenomena, the possible types of critical dynamics correspond to (attractive) fixed points of the *dynamical* RG. This tool allows one to compute the exponents χ and z exactly or approximately.

3.1. The adhesion model

In the cosmological context, the linear regime is identified with small departures from the homogeneous Hubble expansion. The linear dynamics is described by a set of linearized equations for density and velocity. The phenomenological *adhesion model* is the simplest

dynamics describing nonlinear structure formation in cosmology. It relies on the *Zel'dovich approximation* [9], which consists in extrapolating into the nonlinear regime the condition of *parallelism* between velocity and gravity arising in the linear regime. In terms of a rescaled time and a rescaled velocity measuring departures from the Hubble flow, the adhesion model reduces to the *Burgers equation* (originally, an equation for compressible turbulence) [8, 10]:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u}, \tag{4}$$

with $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ by the parallelism assumption, and the mass distribution is obtained from the (rescaled) velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ by means of the continuity equation. Here, $\nu \to 0^+$ is a phenomenological vanishingly small viscosity modelling the coupling to the unresolved small-scale degrees of freedom. The case $\nu = 0$ corresponds actually to the Zel'dovich approximation: the fluid elements move with a constant velocity (in the rescaled variables) along the initial gravitational acceleration, thus effectively neglecting the effect of pressure, viscosity or other small-scale effects. Obviously, singularities arise[3] and the density field diverges. A small, but nonvanishing value of ν regularizes these singularities into *shocks*: it amounts to an *inelastic collision* prescription, such that fluid particles adhere to each other at caustics, which become the walls (pancakes), filaments and nodes that are typical of the large-scale structure.

In the one-dimensional case, velocity and gravity are always parallel. With scale-invariant initial conditions [10], the mass concentrates in shocks located in a *dense set* [4], at which $\boldsymbol{u}$ has discontinuities. This distribution is actually multifractal, namely, a peculiar type of *bifractal*. Furthermore, this bifractal evolves in time: the large shocks grow at the expense of smaller ones, illustrating the bottom-up structure formation typical of cold dark matter.

In three dimensions, the walls, filaments and nodes are definite lower dimensional objects, expected to arise in a generic situation. Therefore, a *naive* picture of this structure consists of a distribution of one-, two- and three-dimensional objects, that is, a trivial example of multifractal distribution, with integer-dimension objects only. However, as the initial velocity field is a non-smooth Gaussian random field [10], the structure produced resembles a self-similar distribution of walls, filaments and nodes that has been dubbed the *cosmic web*. In this 'web' the mass concentrates, in addition to walls, filaments and nodes, in some regions rather than in others (because those objects concentrate there). The cosmic web is a non-trivial multifractal.

3.2. Kardar–Parisi–Zhang equation

The dissipative nature of the Burgers equation (4) is directly related to the coupling to small scales. It seems natural to assume that the dissipation is complemented by a stochastic force (or *noise*). In terms of the velocity potential φ, i.e. $\boldsymbol{u} = -\nabla \varphi$, the stochastic Burgers equation becomes the Kardar–Parisi–Zhang equation[5]:

$$\frac{\partial \varphi}{\partial t} - \frac{1}{2}(\nabla \varphi)^2 = \nu \nabla^2 \varphi + \eta, \qquad \langle \eta(\boldsymbol{x}, t)\eta(\boldsymbol{x}', t')\rangle = D\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t'). \tag{5}$$

The KPZ equation has critical regimes where φ exhibits dynamical scaling and the matter distribution is no longer determined solely by the initial conditions, but is instead the outcome

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[3] The situation is analogous to the formation of *caustics* in geometric optics.

[4] The adjective 'dense' is understood with its mathematical meaning: a set is dense in an interval, say, if in any sub-interval, however small, there are points of the set.

[5] Under this name, this equation is used for the description of surface growth.

of the interplay between the noise and the other terms of the equation. As a general result, invariance under Galilean boosts required by the convective nonlinearity implies $\chi + z = 2$.

In the one-dimensional case, the large-scale dynamics is dominated by a weakly nonlinear (i.e., perturbative) fixed point at which $\chi = 1/2$ and $z = 3/2$. Furthermore, the Fokker–Planck equation for the probability $\Pi[\varphi(x), t]$ associated with the KPZ equation (5) and the Fokker–Planck equation stationary solution are

$$\frac{\partial \Pi}{\partial t} = - \int \mathrm{d}x \, \frac{\delta}{\delta \varphi(x)} [F(\varphi) \Pi] + \frac{D}{2} \int \mathrm{d}x \, \frac{\delta^2 \Pi}{\delta \varphi(x)^2} \Rightarrow \Pi_{\mathrm{stat}} = \exp\left(-\frac{\nu}{2D} \int \mathrm{d}x \, (\partial_x \varphi)^2\right), \quad (6)$$

where $F(\varphi) = \nu \partial_x^2 \varphi + \frac{1}{2} (\partial_x \varphi)^2$. Π_{stat} is a Boltzmann velocity distribution at temperature $kT = 2D/\nu$. Although Π_{stat} is Gaussian, as in the absence of nonlinearity, the value of z is different from the linear equation value ($z_{\mathrm{lin}} = 2$ associated with the diffusion equation). In consequence, the dynamical scaling regime can be considered linear as regards the stationary distribution, but not as regards temporal scaling. Some features involving temporal scaling can be calculated perturbatively with the RG. According to the stationary solution, the shocks predicted by the adhesion model in $d = 1$ disappear at large times as the noise kicks in at a finite temperature T.

The critical dimension of the KPZ equation (5) is $d = 2$, meaning that the nonlinearity is relevant if $d \leqslant 2$. For $d > 2$, the equation becomes perturbatively non-renormalizable, but there are evidences of a scaling strong-coupling regime. In particular, there is an evidence based on the non-perturbative RG [17], a tool that we will introduce in section 4.2. In $d = 3$, the KPZ equation driven by the more general *coloured noise*, $\langle \eta(\boldsymbol{x}, t) \eta(\boldsymbol{x}', t') \rangle \propto |\boldsymbol{x} - \boldsymbol{x}'|^{2\rho - 3}$, has been considered in the cosmological context [18]. There appear weakly nonlinear fixed points with exponents χ, z depending sensitively on the decay exponent ρ of the noise correlator.

A likely failure of the adhesion and KPZ approach is the absence of vorticity imposed by the assumption of parallelism between velocity and gravity. Recently, Antonov [19] applied the RG to the Burgers equation appended with a coloured stochastic source of vorticity. A perturbative fixed point arises where the scaling behaviour is related to the generation of vorticity and depends on the decay exponent of the noise.

4. Approaches based on coarse graining

The KPZ models are rather phenomenological, namely, they miss a first-principle derivation of or a physical argument for the choice of noise correlator, which would make a quantitative prediction available. Now we review briefly more systematic approaches, which are still under development.

4.1. The small-size expansion

The hydrodynamic equations are macroscopic equations, following from microscopic Newtonian mechanics of particles through an averaging process called *coarse graining*. This idea is implemented with the help of a *window function* $W_L(\boldsymbol{r})$, that is, a function that quickly vanishes outside a neighbourhood of the origin of size L; typical examples are the sharp-cutoff (top-hat) window and the Gaussian window. Thus, the coarse-grained mass density field $\varrho_L(\boldsymbol{r})$ is defined as the convolution

$$\varrho_L(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{x} \, W_L(\boldsymbol{r} - \boldsymbol{x}) \varrho(\boldsymbol{x}) \tag{7}$$

with the microscopic mass density field $\varrho(\boldsymbol{x})$. One defines similarly a velocity field $\boldsymbol{v}_L(\boldsymbol{r})$, etc. The exact set of equations for ϱ_L and $\boldsymbol{v}_L$, corresponding to the balance of mass and momentum, contains unknown terms which describe the coupling of the coarse-grained dynamics to the degrees of freedom in scales $< L$. Thus, a constitutive relation is required expressing these terms as functions of the coarse-grained fields.

Recently, the small-size expansion (SSE) has been proposed [20], which relies on the physical hypothesis that the main contribution by the small scales stems precisely from the scales close to L. One finds a gradient expansion like those suggested in the coarse-grain approach to incompressible turbulence. In particular, the phenomenological viscous term in the adhesion model (4) is replaced to lowest order in the expansion by a term proportional to

$$\frac{L^2}{\varrho_L} \left\{ (\nabla \varrho_L \cdot \nabla) \boldsymbol{g}_L - \nabla \cdot \left[\varrho_L \sum_k \left(\frac{\partial \boldsymbol{v}_L}{\partial x_k} \right) \left(\frac{\partial \boldsymbol{v}_L}{\partial x_k} \right) \right] \right\}, \tag{8}$$

where $\boldsymbol{g}_L$ is the coarse-grained gravitational field. This term may act like a drain of kinetic energy in collapsing regions, and the adhesion model can be actually recovered under stronger dynamical assumptions like parallelism of $\boldsymbol{v}_L$ and $\boldsymbol{g}_L$. However, unlike the adhesion model, this term also behaves as a source of vorticity.

4.2. The exact renormalization group

The coarse-grained variables change with the coarse-graining length L and so does the probability distribution, e.g., of the density $\mathcal{P}_L[\varrho]$ (in Fourier space):

$$\frac{\partial}{\partial L} \mathcal{P}_L[\varrho] = \frac{\mathrm{d} \ln \tilde{W}_L^2}{\mathrm{d} L} \frac{\delta}{\delta \varrho} (\varrho \mathcal{P}_L[\varrho]) + \frac{1}{2} \frac{\mathrm{d} \tilde{W}_L^2}{\mathrm{d} L} P(k) \frac{\delta^2}{\delta \varrho^2} \mathcal{P}_L[\varrho], \tag{9}$$

where $\tilde{W}_L(k)$ is the Fourier transform of the window function, defined in equation (7), and $P(k)$ is the large-scale power spectrum of the density fluctuations, i.e., the Fourier transform of the correlation $\xi(r)$ introduced in section 2.1. The differential equation (9), describing evolution with the scale L, is an *exact RG* equation for $\mathcal{P}_L$ that constitutes a sort of Fokker–Planck equation for L evolution [21].

Exact RG equations have been amply used in high-energy and statistical physics [11, 22]. In those applications, the exact RG is often described in terms of the effective average action [23]. In our context, the unusual form given by equation (9) is more convenient, but it has not been studied in the astrophysical literature yet (see [21]). However, it has been noted that the one-point density probability distribution $p_L(\varrho)$ satisfies a diffusion equation which can be used to find the Press-Schechter mass function of collapsed objects [24]. This diffusion equation is connected with equation (9).

4.2.1. Time evolution and RG.

Peebles [25] noted that the temporal variable, in some cosmological solutions (scaling solutions), plays the role of a scaling parameter and proposed a type of renormalization in which the time evolution can be undone by a redefinition of the space scale and the number and mass of particles. Therefore, Peebles assumes a relationship between time evolution and evolution under the change of scale, in the same fashion as dynamical scaling. That relationship admits a fuller formulation in terms of the exact RG: equation (9) can be compared with a Fokker–Planck equation for time evolution, like the one for the KPZ equation found in section 3.2 [21].

5. Conclusions

From the observational standpoint, scaling in the large-scale structure is well justified, but the measures are not sufficiently accurate yet to determine many details. The scale of homogeneity, which has been the subject of much controversy, seems to be in the range 10–20 Mpc/h. There seems to be no point in trying to determine a definite value of the fractal dimension. Rather, the distribution fits a multifractal, so its scaling properties are given by its multifractal spectrum, which can already be found with the help of N-body simulations [7]. The determination of fine morphological features requires further analysis of the galaxy distribution as well as N-body simulations, with appropriate tools such as statistical moments or more sophisticated tools (Minkowski functionals, etc). Among these morphological features are the voids. Scaling of voids is beginning to be observed, but deeper studies of voids will depend on improvement on their definition and, hence, detection.

From the theoretical standpoint, scaling provides us with a handle in an otherwise almost intractable problem of nonlinear dynamics. We have seen that dynamical scaling is indeed sufficiently powerful to draw a convincing picture of structure formation in one dimension, with the help of perturbation theory and the renormalization group. However, the realistic three-dimensional case may demand non-perturbative tools which are rather complex and are still being developed.

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